HALE
SCHOOL

Semester 2 Examination 2016
Question/Answer booklet

Year 12
MATHEMATICS SPECIALIST
Book 1 of 2
Section One
(Calculator-free)

ANSWERS

Circle teacher's name Mr Hill Mr Lau

Time allowed for this section
Reading time before commencing work: 5 minutes
Working time for section: 50 minutes

Material required/recommended for this section
To be provided by the supervisor
This Question/Answer booklet for Section One, and a separate formula sheet which may also be used for Section Two.

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, highlighter, eraser, ruler

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this examination

	Number of questions available	Number of questions to be attempted	Working time (minutes)	Marks available	Percentage of Exam
Section One Calculator—free	**8**	**8**	**50**	**53**	**35%**
Section Two Calculator—assumed	13	13	100	99	65%
			Total marks	152	100%

Instructions to candidates

1. Answer all the questions in the spaces provided.

2. Spare answer pages are provided at the end of this booklet. If you need to use them, indicate in the original answer space where the answer is continued i.e. give the page number.

3. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Correct answers given without supporting reasoning may not be allocated full marks. Incorrect answers given without supporting reasoning cannot be allocated any marks. If you repeat an answer to any question, ensure that you cancel the answers you do not wish to have marked.

4. It is recommended that you **do not use pencil** except in diagrams.

Question 1 **(5 marks)**

(a) Evaluate $(2 - i)(1 - 2i)$, leaving your answer in polar form. (2 marks)

$(2-i)(1-2i)$
$= -5i$
$= 5\,\text{cis}\left(-\frac{\pi}{2}\right)$

Multiplies out correctly ✓
Converts to polar form ✓

(b) If $z_1 = \text{cis}\,\frac{\pi}{6}$, $z_2 = r\,\text{cis}\,\theta$ and $z_1 z_2$ is a real number, determine the value(s) of θ where $-\pi < \theta \le \pi$. (3 marks)

$z_1 z_2 = r\,\text{cis}\left(\frac{\pi}{6} + \theta\right)$

$z_1 z_2$ is a real no. $\Rightarrow$ $\frac{\pi}{6} + \theta = 0$ or π

$\therefore\ \theta = -\frac{\pi}{6}$ or $\frac{5\pi}{6}$

Determines the argument for z_1z_2 ✓
Equates the argument = 0 or π ✓
States the answers ✓

Question 2 **(7 marks)**

(a) Two random samples of 9 students from Year 12 Hale School are taken. The following table shows their brief summary of statistics of their heights.

	Mean (cm)	Standard deviation (cm)
Sample 1	177.5	10.5
Sample 2	179.2	9.4

Briefly explain why the two sample means are expected to be different. (2 marks)

The sample mean $\bar{X}$ is a random variable whose value is subject to change due to chance.

Explains that $\bar{X}$ is a random variable	✓
States that the value of $\bar{X}$ changes due to chance	✓

(b) The sampling distribution of the heights of random samples of 9 students from Year 12 Hale School has a mean of 178.4 cm and a standard deviation of 8.2 cm.

(i) Find the mean and standard deviation of **all** Year 12 Hale students' height. (3 marks)

population mean $\mu = E(\bar{X}) = 178.4$ cm

standard deviation $\sigma = s \times \sqrt{n} = 8.2 \times \sqrt{9} = 24.6$ cm

States the population mean	✓
Uses the correct formula for calculating σ	✓
States the population standard deviation	✓

(ii) Is it valid to assume that the sampling distribution of the heights is approximately normal? Explain. (2 marks)

NO

Sample size 9 is less than 30 (and population distribution unknown).

States not valid	✓
Provides a valid explanation	✓

Question 3 **(5 marks)**

(a) Determine the value of c if the three equations $\begin{cases} x + 2y - z = 0 \\ 2x - y + z = -1 \\ -x + y + cz = 1 \end{cases}$ have **no** solutions. (4 marks)

$x + 2y - z = 0 \quad \ldots\ldots (1)$
$2x - y + z = -1 \quad \ldots\ldots (2)$
$-x + y + cz = 1 \quad \ldots\ldots (3)$

$(1) + (3): \quad 3y + (c-1)z = 1 \quad \ldots\ldots (4)$
$(2) + 2\times(3): \quad y + (2c+1)z = 1 \quad \ldots\ldots (5)$

$3\times(5) - (4): \quad (5c+4)z = 2$

no solutions $\Rightarrow \quad c = -\dfrac{4}{5}$

Eliminates one variable ✔
Eliminates another variable ✔
Sets the coefficient of the remaining variable to zero ✔
States the answer ✔

(b) Given the answer in (a), give a geometric interpretation of the three planes in (a). (1 mark)

The three planes intersect in parallel lines.

States parallel intersection lines ✔

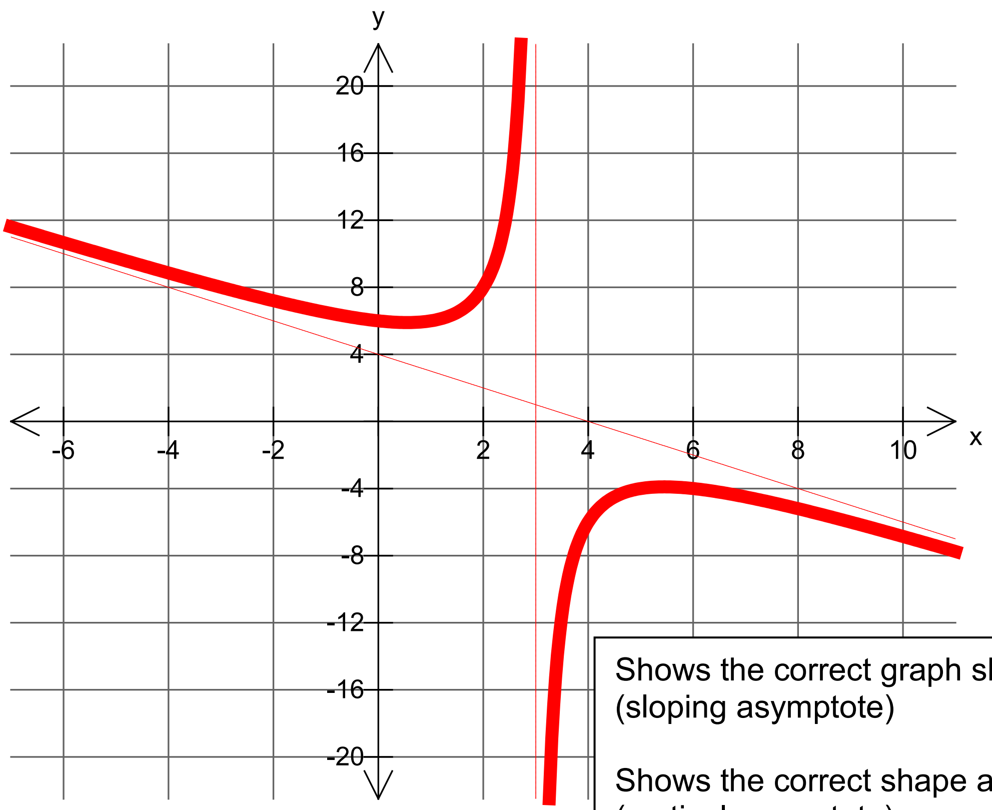
Question 4 **(6 marks)**

(a) Simplify the function $y = \frac{-x^2+7x-18}{x-3}$ to the form $y = (ax+b) - \frac{6}{x-3}$. (2 marks)

$$y = \frac{-x^2+7x-18}{x-3}$$
$$= \frac{(x-3)(-x+4)-6}{x-3}$$
$$= (-x+4) - \frac{6}{x-3}$$

Shows the correct value of a ✓
Shows the correct value of b ✓

(b) Hence sketch the graph of the function $y = \frac{-x^2+7x-18}{x-3}$. (4 marks)



Shows the correct graph shape as x → ±∞ (sloping asymptote) ✓
Shows the correct shape as x → 3 (vertical asymptote) ✓
Shows the correct y-intercept ✓
Shows a local min and a local max ✓

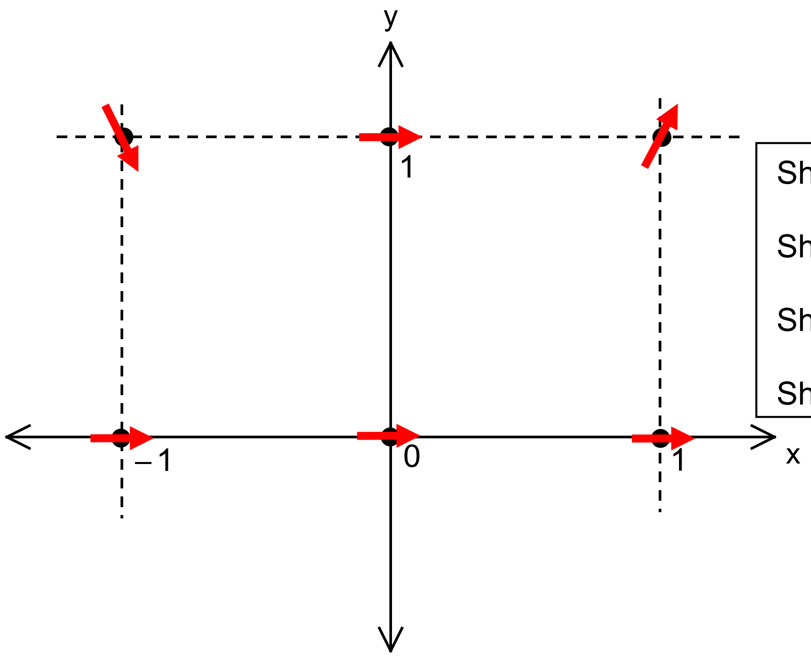
Question 5 **(8 marks)**

(a) Solve $\dfrac{dy}{dx} = 4x\sqrt{y}$ if y = 1 when x = -1. (4 marks)

$$\frac{dy}{dx} = 4x\sqrt{y}$$
$$\Rightarrow \quad \frac{dy}{\sqrt{y}} = 4x\,dx$$
$$\Rightarrow \quad 2\sqrt{y} = 2x^2 + c$$
$$\Rightarrow \quad 2\sqrt{1} = 2(-1)^2 + c$$
$$\Rightarrow \quad c = 0$$
$$\Rightarrow \quad \sqrt{y} = x^2$$

Separates the variables
Integrates correctly
Uses (1, -1) to solve for c
States the answers

(b) Sketch the slope fields of $\dfrac{dy}{dx} = 4x\sqrt{y}$ for the 6 points indicated below. (4 marks)



Shows horizontal lines at x = 0
Shows horizontal lines at y = 0
Shows positive slope at (1, 1)
Shows negative slope at (-1, 1)

Question 6 **(11 marks)**

Find the following integrals.

(a) $\int \cos^2 2x \, dx$ (3 marks)

$$\int \cos^2 2x \, dx = \int \frac{1}{2}(\cos 4x + 1) \, dx$$
$$= \int \frac{1}{2}\cos 4x \, dx + \int \frac{1}{2} dx$$
$$= \frac{1}{8}\sin 4x + \frac{1}{2}x + c$$

Uses the correct trig identity	✓
Integrates cos 4x correctly	✓
States the correct answer, including the constant c	✓

(b) $\int \frac{3x+7}{x^2+3x-4} dx$ (3 marks)

$$\int \frac{3x+7}{x^2+3x-4} dx = \int \left(\frac{1}{x+4} + \frac{2}{x-1}\right) dx$$
$$= \ln|x+4| + 2\ln|x-1| + c$$

Uses partial fractions	✓
Obtains correct numerators	✓
States the answer	✓

(c) $\int \frac{2+\ln x}{x} dx$ where $x > 0$ (2 marks)

$$\int \frac{2+\ln x}{x} dx = \int \frac{1}{x}(2+\ln x) dx$$
$$= \frac{(2+\ln x)^2}{2} + c$$
$$[= \frac{4\ln x + (\ln x)^2}{2} + C]$$

Recognizes that the integral is of the form $\int f'(x)\, f(x)\, dx$ ✓
Integrates correctly ✓

(d) $\int \frac{-1}{1+e^x} dx$ (Hint: $a^b = \frac{1}{a^{-b}}$) (3 marks)

$$\int \frac{-1}{1+e^x} dx = \int \frac{-1}{1+\frac{1}{e^{-x}}} dx$$
$$= \int \frac{-e^{-x}}{e^{-x}+1} dx$$
$$= \ln|e^{-x}+1| + c$$

Writes e^x as $1/e^{-x}$ ✓
Writes the integrand in the form $f'(x)/f(x)$ ✓
States the answer ✓

Question 7 **(5 marks)**

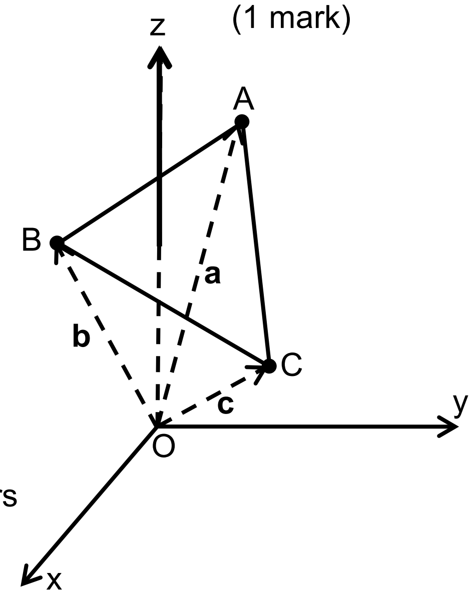
(a) Give a geometric interpretation of the magnitude of $\mathbf{a}\times\mathbf{b}$ regarding the parallelogram spanned by the vectors **a** and **b**. (1 mark)

$\mathbf{a}\times\mathbf{b} = |\mathbf{a}|\times|\mathbf{b}|\times\sin\theta$
So the magnitude of $\mathbf{a}\times\mathbf{b}$ represents the area of the parallelogram spanned by **a** and **b**.

Gives the correct interpretation ✓

(b) Consider 3 points A, B and C with corresponding position vectors **a**, **b** and **c** in space as shown below. The points O, A, B and C form a tetrahedron.

(i) Express **AB** and **AC** in terms of **a**, **b** and/or **c**. (1 mark)



$\mathbf{AB} = \mathbf{b} - \mathbf{a}$ and $\mathbf{AC} = \mathbf{c} - \mathbf{a}$

States the answers ✓

(ii) The area vector of a face is a vector perpendicular to the face, pointing **outwards**, whose magnitude is the area of the face.
Show that the sum of the outward pointing area vectors of the 4 faces of the tetrahedron equals the zero vector. (3 marks)

Sum of pointing area vectors

$$= \frac{1}{2}[\mathbf{b}\times\mathbf{a} + \mathbf{c}\times\mathbf{b} + \mathbf{a}\times\mathbf{c} + (\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})]$$

$$= \frac{1}{2}(\mathbf{b}\times\mathbf{a} + \mathbf{c}\times\mathbf{b} + \mathbf{a}\times\mathbf{c} + \mathbf{b}\times\mathbf{c} - \mathbf{b}\times\mathbf{a} - \mathbf{a}\times\mathbf{c})$$

$$= \mathbf{0}$$

Writes down the area any face using the correct cross product ✓

Multiplies out correctly ✓

Shows that the sum equals the zero vector ✓

Question 8 **(6 marks)**

A model for a population, P, of numbats is $P = \dfrac{900}{3+2e^{-\frac{t}{4}}}$, where t is the time in years from the start of 2015.

(a) What is the initial population? (1 mark)

$$P = \frac{900}{3+2} = 180$$

States the answer ✓

(b) What is the predicted long term population? (1 mark)

$$P = \frac{900}{3} = 300$$

States the answer ✓

(c) Show **clearly** that P satisfies the differential equation $\dfrac{dP}{dt} = \dfrac{P}{4}(1 - \dfrac{P}{300})$. (4 marks)

$$P = \frac{900}{3+2e^{-\frac{t}{4}}} \quad \Rightarrow \quad e^{-\frac{t}{4}} = \frac{450}{P} - \frac{3}{2}$$

$$\Rightarrow \quad -\frac{e^{-\frac{t}{4}}}{4} = -\frac{450}{P^2}\frac{dP}{dt}$$

$$\Rightarrow \quad \frac{dP}{dt} = \frac{P^2}{4\times 450}e^{-\frac{t}{4}}$$

$$\Rightarrow \quad \frac{dP}{dt} = \frac{P^2}{4\times 450}\left(\frac{450}{P} - \frac{3}{2}\right)$$

$$\therefore \quad \frac{dP}{dt} = \frac{P}{4}\left(1 - \frac{300}{P}\right)$$

Expresses $e^{-t/4}$ in terms of P ✓
Differentiates to get dP/dt ✓
Uses $e^{-t/4}$ in terms of P to express dP/dt in terms of P ✓
Simplifies correctly ✓

Additional working space

Question number(s): ________________

HALE SCHOOL

Semester 2 Examination 2016

Question/Answer booklet

Year 12
MATHEMATICS SPECIALIST
Book 2 of 2
Section Two
(Calculator-assumed)

Answers

Circle teacher's name Mr Hill Mr Lau

Time allowed for this section

Reading time before commencing work: 10 minutes
Working time for section: 100 minutes

Material required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet for Section Two. Candidates may use the separate formula sheet from Section One.

To be provided by the candidate

Standard items: pens, pencils, pencil sharpener, highlighter, eraser, ruler
Special items: drawing instruments, templates, notes on up to two unfolded sheets of A4 paper, and up to three calculators, approved for use in this examination.

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Question 9 **(6 marks)**

Suppose that the heights of students at Hale School follow a normal distribution. It is found that a large random sample of Hale students has a mean of 172 cm with a 6 cm margin of error at a confidence level of 90%.

(a) Write down the confidence interval for estimating the mean height of Hale School students. Interpret your answer. (2 marks)

Confidence interval: [166, 178]

We are 90% confident that the interval [166, 178] contains the true mean height of Hale students.

Or About 90% of repeated random samples will contain the true mean height of Hale students.

Gives the correct interval ✓

Provides a valid interpretation ✓

(b) What would be the change in the confidence interval (increase, decrease or no change) if the sample size increases? Explain. (2 marks)

Confidence interval decreases

Randomness of the data is smoothened off with a larger sample size, so margin of error decreases and hence confidence interval decreases.

States the CI decreases ✓

Provides a valid interpretation ✓

(c) What would be the change in the confidence interval if the confidence level is set at 95% instead of 90%? Explain. (2 marks)

Confidence interval increases

At 95% confidence level, the margin of error would increase. Hence the confidence interval increases.

States the CI increases ✓

Provides a valid interpretation ✓

Question 10 **(7 marks)**

Consider the functions $f(x) = \ln(5e - x)$ and $g(x) = e^x + 4e$.

(a) Give an example of x such that $f(g(x))$ is **not** defined. (1 mark)

$x = 1$ (any $x \ge 1$)	States the answer ✓

(b) (i) State the largest restricted domain of $g(x)$ such that $f(g(x))$ is a function. (2 marks)

Restricted $D_g = \{x : x < 1\}$	Gives the correct upper bound ✓ Uses the correct inequality sign ✓

Consider the restricted domain of $g(x)$ in part (b) (i).

(ii) Show that $f(g(x))$ is a one-to-one function. (4 marks)

$y = f(g(x)) = \ln(e - e^x)$ $\frac{dy}{dx} = \frac{-e^x}{e - e^x} < 0$ for $x < 1$ $\Rightarrow$ $f(g(x))$ is a strictly decreasing function. $\Rightarrow$ $f(g(x))$ is a one-to-one function.	Shows the correct composite function ✓ Differentiates correctly ✓ Shows that dy/dx is less than 0 ✓ Provides a reason for 1-1 function ✓

OR

$y = f(g(x)) = \ln(e - e^x)$ $f(g(y)) = f(g(x))$ $\ln(e - e^y) = \ln(e - e^x)$ ($\ln x$ 1-to-1) $e - e^y = e - e^x$ $-e^y = -e^x$ (e^x 1-to-1) $x = y$ *if* $f(g(y)) = f(g(x)) \to x = y$ then the contrapositive is true $x \ne y \to f(g(y)) \ne f(g(x))$ $\therefore f(g(x))$ is 1-to-1 .	Shows the correct composite function ✓ Equates f(g(y))=f(g(x) ✓ With reasons establishes x=y ✓ Provides a reason for 1-1 function ✓

Question 11 **(8 marks)**

Random samples of size 36 are drawn from a large population of ball bearings.
A particular sample has a mean of 22.40 g, with a standard deviation of 0.72 g.

(a) Estimate the standard deviation of the sampling distribution of the mean. (2 marks)

$$\sigma_{\bar{x}} \approx \frac{0.72}{\sqrt{36}} = 0.12 \text{ g}$$

Uses the correct formula ✓
States the answer ✓

Consider a 96% confidence interval for the population mean.

(b) State the values (correct to 4 decimal places) of the standard normal variable corresponding to 96% confidence interval. (1 mark)

CAS: $z = [-2.0537,\ 2.0537]$

States the answer ✓

(c) Determine the 96% confidence interval for the population mean. (2 marks)

Confidence Interval:
$$[22.40 - 2.0537 \times 0.12,\ 22.40 + 2.0537 \times 0.12]$$
$$= [22.15,\ 22.65]$$

Uses the correct interval formula ✓
States the answer ✓

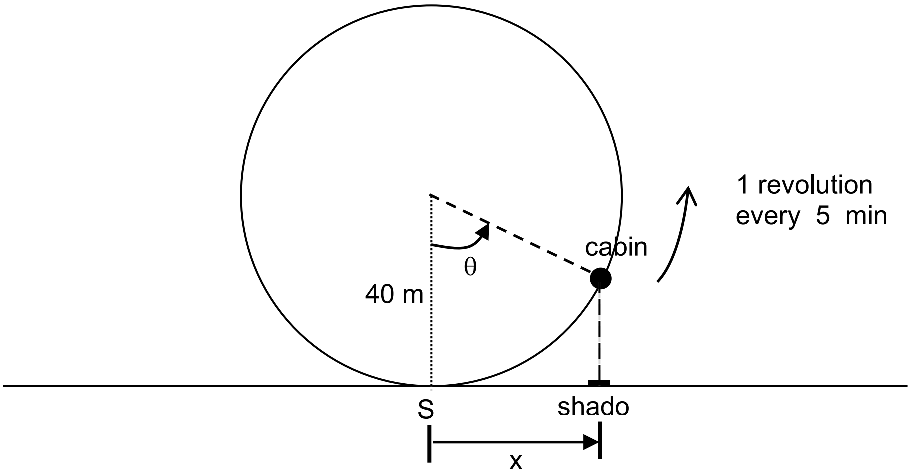
(d) Assume that the population standard deviation of ball bearings can be estimated by 0.72 g, determine how large a sample must be taken in order to be 96% confident that the error in the estimation of population mean will not exceed 0.15 g. (3 marks)

$$\text{error} = 2.0537 \times \frac{0.72}{\sqrt{n}} \leq 0.15$$
$$\Rightarrow \quad n \geq 97.18$$
$$\therefore \quad n = 98$$

Uses the correct error formula ✓
Solves the inequality correctly ✓
States the answer ✓

Question 12 **(8 marks)**

Laura is travelling on a Ferris wheel of radius 40 metres, that is turning at a constant angular speed of one revolution every 5 minutes. Initially, Laura's cabin is at ground level, at point S. The sun is directly overhead and casts a shadow on the ground directly below Laura's cabin.



Let θ be the angle that the radius joining the centre of the Ferris wheel to the cabin makes with the downward vertical, and
x be the **horizontal** displacement of the shadow from the point S on the ground.

(a) Express θ in terms of time t (in min). (2 marks)

$$\frac{d\theta}{dt} = \frac{2\pi}{5}$$

$$t = 0,\ \theta = 0 \quad \Rightarrow \quad \theta = \frac{2\pi}{5}t$$

Determines $d\theta/dt$	✓
States the answer	✓

(b) Show that the **shadow** moves with simple harmonic motion. (3 marks)

$$x = 40\sin\frac{2\pi}{5}t$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{2\pi}{5} \times 40\cos\frac{2\pi}{5}t$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} = -(\frac{2\pi}{5})^2 \times 40\sin\frac{2\pi}{5}t = -(\frac{2\pi}{5})^2 x$$

$\therefore$ The shadow describes SHM

Determines the displacement equation	✓
Differentiates x twice to obtain the acceleration	✓
Identifies the SHM eq	✓

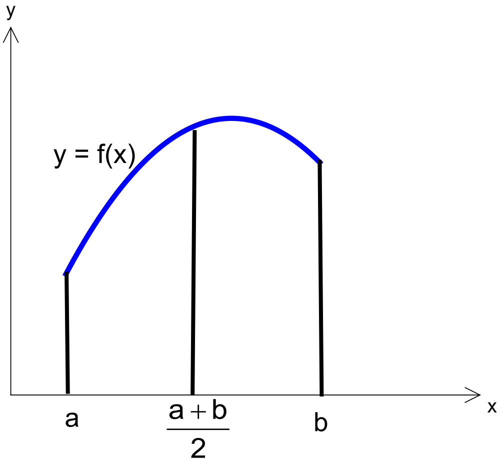
(c) Find the exact speed at which the shadow is moving horizontally, when the cabin is 20 m above the ground. (3 marks)

$\text{height} = 20\text{ m} \quad \Rightarrow \quad \theta = \frac{\pi}{3}$	Determines θ for h = 20 m ✓
and so $t = \frac{5}{6}$ min $\quad (\frac{1}{6}$ of a cycle$)$	Determines the corresponding time ✓
$\Rightarrow \quad \frac{dx}{dt} = \frac{2\pi}{5} \times 40\cos\frac{2\pi}{5}(\frac{5}{6})$ $\therefore \quad \text{speed} = 8\pi \text{ m/min}$	Uses the velocity equation to determine the speed ✓

Question 13 **(9 marks)**

The area A under the graph of $y = f(x)$ from $x = a$ to $x = b$ can be approximated by

$$A \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right].$$



The above method is used to estimate the area under the graph of $y = x \ln x$ from $x = 2$ to $x = 3$.

Use 4 strips to approximate the area and give your answers correct to 4 decimal places for parts (a) and (b), if necessary.

(a) Complete the following table. (2 marks)

x	2	2.25	2.5	2.75	3
f(x)	1.3863	1.8246	2.2907	2.7819	3.2958

Shows correct x values ✓
Evaluates y values ✓

(b) Show **clearly** how you use the above formula to obtain an estimate of the required area. (4 marks)

$$\begin{aligned}\text{Area} &= \int_2^3 f(x)dx \\ &= \int_2^{2.5} f(x)dx + \int_{2.5}^3 f(x)dx \\ &\approx \frac{3-2}{12}(1.3863+4\times1.8246+2.2907) \\ &\quad + \frac{3-2}{12}(2.2907+4\times2.7819+3.2958) \\ &= 2.3075\end{aligned}$$

Expresses the area as the sum of two areas with correct boundaries ✓
Calculates width correctly ✓
Uses the given formula correctly ✓
States the answer ✓

(c) State an approximation of the area correct to **6** decimal places. Provide evidence for your estimation. (2 marks)

$n = 30: \quad A = 2.3074609388\ldots$

$n = 32: \quad A = 2.3074609386\ldots$

$\therefore \quad A \approx 2.307461 \quad (6 \text{ d.p.})$

Provides an evidence ✓
States the answer ✓

(d) State the **type** of function $f(x)$ such that the formula

$$A \approx \frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)]$$

actually gives an exact answer. (1 mark)

Any polynomial function of degree up to 3.

States the answer ✓

Question 14 **(6 marks)**

Let $z = x + yi$ be a complex number where x and y are real numbers.

(a) Show that $\dfrac{z-2i}{z+2i} = \dfrac{(x^2+y^2-4)-4xi}{x^2+(y+2)^2}$. (2 marks)

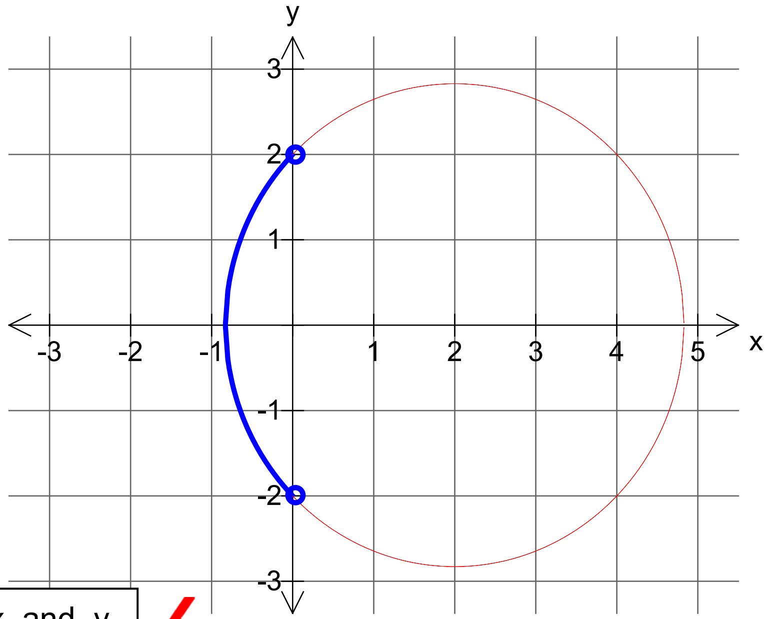
$$\frac{z-2i}{z+2i} = \frac{x+(y-2)i}{x+(y+2)i} \times \frac{x-(y+2)i}{x-(y+2)i}$$
$$= \frac{(x^2+y^2-4)+[x(y-2)-x(y+2)]i}{x^2+(y+2)^2}$$
$$= \frac{(x^2+y^2-4)-4xi}{x^2+(y+2)^2}$$

Realises the expression ✓

Simplifies correctly ✓

(b) Sketch the locus of z such that $\arg\left(\dfrac{z-2i}{z+2i}\right) = \dfrac{3\pi}{4}$. (4 marks)

$$\arg\left(\frac{z-2i}{z+2i}\right) = \frac{3\pi}{4}$$
$$\Rightarrow \quad \tan^{-1}\left(\frac{-4x}{x^2+y^2-4}\right) = \frac{3\pi}{4}$$
$$(x < 0 \text{ and } x^2+y^2 < 4)$$
$$\Rightarrow \quad \frac{-4x}{x^2+y^2-4} = -1$$
$$\Rightarrow \quad x^2-4x+y^2 = 4$$
$$\Rightarrow \quad (x-2)^2+y^2 = 8$$



Writes the argument in terms of x and y ✓

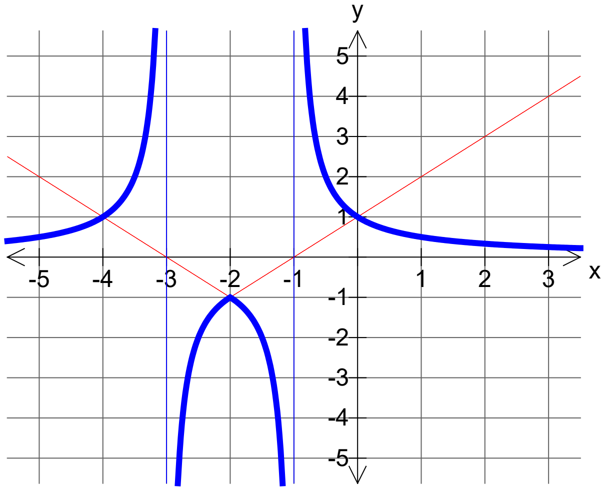
Sets up the algebraic equation ✓

Simplifies to get the circle equation ✓

Shows the correct arc ✓

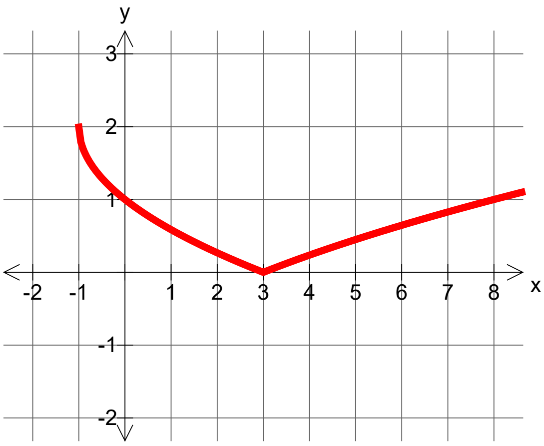
Question 15 **(7 marks)**

(a) Given the graph of the function y = f(x) as shown below, sketch the graph of its reciprocal function $y = \frac{1}{f(x)}$. (3 marks)



Shows vertical asymptotes at -3 and 1 ✓
Passes through (-2, -1) ✓
Shows a reasonable shape ✓

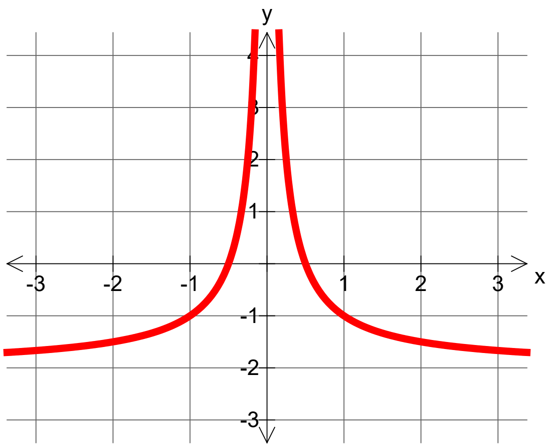
(b) Determine the function y = g(x), where g(x) is a square-root function, such that the graph of y = | g(x) | is as shown below. (2 marks)



$g(x) = \sqrt{x+1} - 2$
or $g(x) = -\sqrt{x+1} + 2$

Gives the correct horizontal translation of g(x) ✓
Gives the correct vertical translation of f(x) ✓

(c) Determine the function y = h(x), where h(x) is a reciprocal function, such that the graph of y = h(|x|) is as shown below. (2 marks)

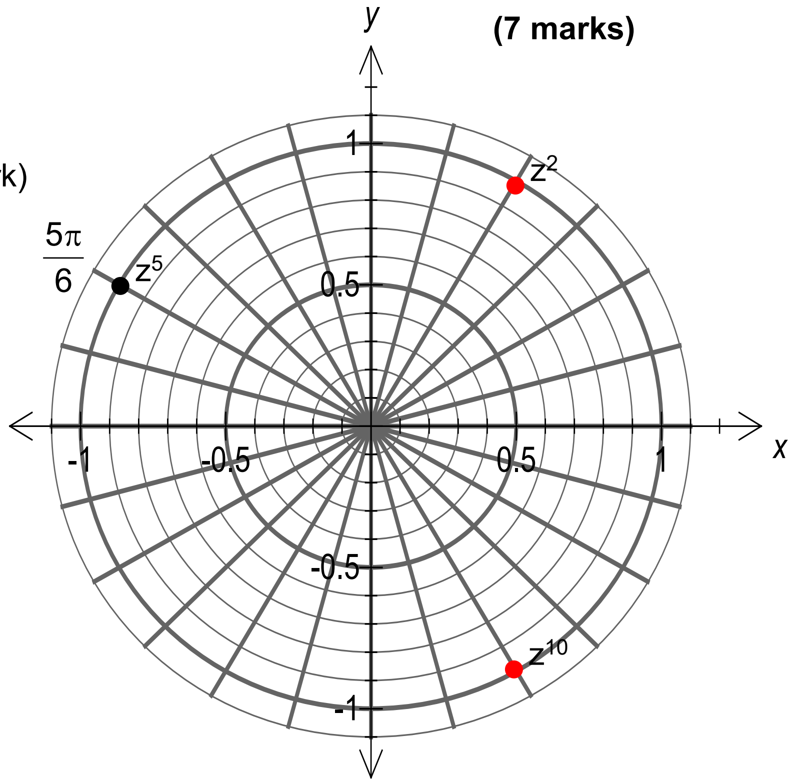


$h(x) = \frac{1}{x} - 2$

Gives the term 1/x ✓
Gives the correct vertical translation of h(x) ✓

Question 16 **(7 marks)**

A complex number z^5 is shown on the right.



(a) Express z^5 in polar form. (1 mark)

$$z^5 = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$$

States the answer ✓

(b) On the above set of axes, plot the following complex numbers. Label them clearly. (3 marks)

(i) z^2

(ii) z^{10}

Shows correct modulus ✓

Shows correct arguments ✓ ✓

(c) Determine n such that $z^5 = z^n$ where n ($\neq 5$) is a positive integer. (3 marks)

A possible response :

$$z^5 = \text{cis}\frac{5\pi}{6} \quad \Rightarrow \quad z = \text{cis}\frac{\pi}{6}$$

And $z^{12} = \text{cis } 2\pi$

$$z^{17} = z^5 \times z^{12} = \text{cis } (\frac{5\pi}{6} + 2\pi) = z^5$$

$\therefore \quad n = 17$

Solves for z ✓

Shows z^{12} represents a complete revolution ✓

Multiplies z^5 and z^{12} to get z^5 and hence states the answer ✓

Question 17 **(9 marks)**

A line ℓ passes through the z-axis at $z = 2$ and is parallel to the y-axis.

(a) State the vector equation of ℓ. (2 marks)

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Shows that ℓ passes through (0, 0, 2) ✓

Shows that ℓ's direction is parallel to (0, k , 0) ✓

(b) Find the vector equation of the plane π that contains ℓ and passes through (1, 0, 0). (3 marks)

Let $\mathbf{n}$ be a normal vector of π

Then $\mathbf{n} \perp \begin{pmatrix} 1-0 \\ 0-0 \\ 0-2 \end{pmatrix}$ and $\mathbf{n} \perp \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\Rightarrow$ $\mathbf{n}$ is given by $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

So $\pi :$ $\mathbf{r} \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = 2$

Indicates that π is $\perp$ to both (1, 0, -2) and (0, 1, 0) ✓

Uses cross product to find $\mathbf{n}$ ✓

States the answer ✓

(c) The plane π in part (b) touches a sphere whose centre is at (0, 0, 0). Find the Cartesian equation of the sphere. (4 marks)

radius $= \perp$ distance from (0, 0, 0) to π

$\mathbf{r} \cdot \mathbf{n} = |\mathbf{r}| \times |\mathbf{n}| \times \cos\theta = 2$

$\perp$ distance $= |\mathbf{r}| \times \cos\theta$

$\Rightarrow$ radius $= \dfrac{\mathbf{r} \cdot \mathbf{n}}{|\mathbf{n}|}$

$\Rightarrow$ radius $= \dfrac{2}{\sqrt{5}} = \dfrac{2\sqrt{5}}{5}$

So sphere equation: $x^2 + y^2 + z^2 = \dfrac{4}{5}$

Shows radius = $\perp$ distance ✓

Uses $r \cos\theta$ as $\perp$ distance ✓

Determines the radius ✓

States eqn of sphere ✓

Question 18 **(10 marks)**

The position of a particle P is defined by the vector function

$$\mathbf{r}(t) = (-1 + 3\sec bt)\,\mathbf{i} + (2 + 4\tan bt)\,\mathbf{j}$$

where $b > 0$ and $0 \le t \le \pi$.

(a) Determine the Cartesian equation corresponding to the vector equation. (3 marks)

$\mathbf{r}(t) = (-1 + 3\sec bt)\mathbf{i} + (2 + 4\tan bt)\mathbf{j}$

$\Rightarrow \begin{cases} x = -1 + 3\sec bt \\ y = 2 + 4\tan bt \end{cases}$

$\Rightarrow (\frac{x+1}{3})^2 - (\frac{y-2}{4})^2 = 1$

Equates corresponding components	✓
Uses $\sec^2 x = \tan^2 x + 1$	✓
States the answer	✓

(b) Find the velocity vector of the particle P at time t. (1 mark)

$\mathbf{r}(t) = (-1 + 3\sec bt)\mathbf{i} + (2 + 4\tan bt)\mathbf{j}$

$\Rightarrow \mathbf{v}(t) = (3b\tan bt\sec bt)\mathbf{i} + (4b\sec^2 bt)\mathbf{j}$

Differentiates **r**(t) correctly	✓

(c) Particle P is moving vertically when $t = \frac{\pi}{2}$. Determine the minimum value of b. (3 marks)

Moving vertically when $t = \frac{\pi}{2}$

$\Rightarrow v_x(\frac{\pi}{2}) = 0$

$\Rightarrow 3b\tan b(\frac{\pi}{2})\sec b(\frac{\pi}{2}) = 0$

$\Rightarrow \tan b(\frac{\pi}{2}) = 0 \quad (\because \; b \ne 0 \text{ and } \sec bt \ne 0)$

$\Rightarrow b_{min} = 2$

Sets $v_x(\pi/2) = 0$	✓
Sets $\tan b(\pi/2) = 0$ with reasons	✓
States the answer	✓

A second particle Q starts at the same instant as P and moves with position vector given by $\mathbf{r}(t) = (-1+t)\,\mathbf{i} + (5-t)\,\mathbf{j}$.

(d) Assuming the minimum value of b in part (c), determine whether P and Q collide or their paths cross. State the time when the incident happens. (3 marks)

$\mathbf{r}_P(t) = (-1+3\sec 2t)\mathbf{i} + (2+4\tan 2t)\,\mathbf{j}$

$\mathbf{r}_Q(t) = (-1+t)\mathbf{i} + (5-t)\,\mathbf{j}$

$$\Rightarrow \begin{cases} -1+3\sec 2t = -1+T \\ 2+4\tan 2t = 5-T \end{cases} \quad \text{and} \quad \begin{cases} t = T & \text{if they collide} \\ t \neq T & \text{if their paths cross} \end{cases}$$

Using CAS: $t = 0$ or π and $T = 3$

So their paths cross when $t = 3$ or π.

Equates corresponding components	✓
Solves correctly for the times	✓
States their paths cross when t = 3 (or $t = \pi$)	✓

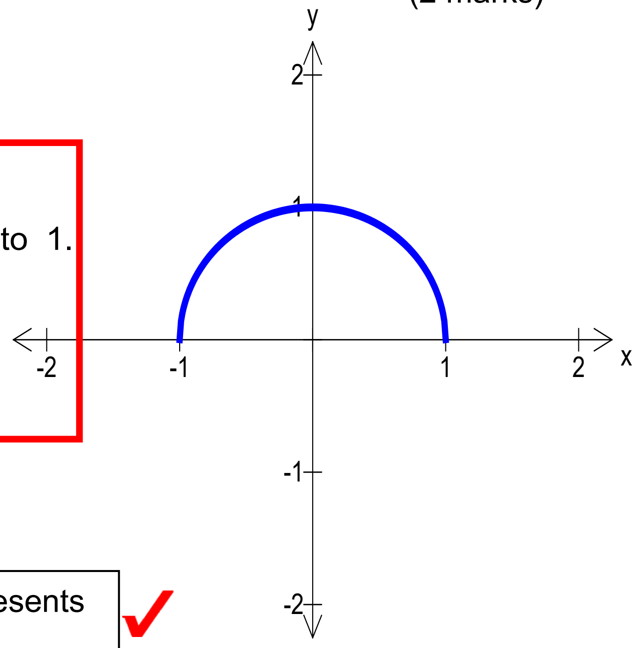
Question 19 **(9 marks)**

(a) By considering the **geometric** interpretation of $\int_{-1}^{1}\sqrt{1-x^2}\,dx$ **explain** why $\int_{-1}^{1}\sqrt{1-x^2}\,dx = \frac{1}{2}\pi$. (2 marks)

$\int_{-1}^{1}\sqrt{1-x^2}\,dx$ represents the area under the unit semi-circle $y=\sqrt{1-x^2}$ from -1 to 1.

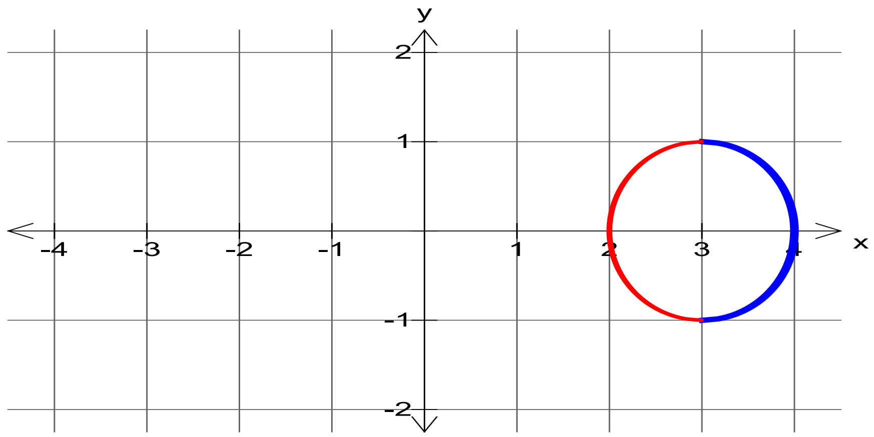
So $\int_{-1}^{1}\sqrt{1-x^2}\,dx = \frac{1}{2}\pi(1)^2$

$= \frac{1}{2}\pi$



States that the integral represents the area under a semi-circle ✓
States the radius ✓

(b) A circular disc with equation $(x - 3)^2 + y^2 = 1$ is rotated about the **y-axis**. The solid obtained is a torus (a donut shaped solid).
Find, by integration, the exact volume of the solid. (7 marks)



Consider that the solid is made up of many thin horizontal rings.

For the ring of thickness δy at a distance y from the x-axis

$x_1 = 3 - \sqrt{1-y^2}$ and $x_2 = 3 + \sqrt{1-y^2}$

Then its volume $\delta V = \pi[(3+\sqrt{1-y^2})^2 - (3-\sqrt{1-y^2})^2]\delta y$

$\Rightarrow \quad \delta V = \pi[9 + 6\sqrt{1-y^2} + (1-y^2) - 9 + 6\sqrt{1-y^2} - (1-y^2)]\delta y$

$\Rightarrow \quad \delta V = 12\pi\sqrt{1-y^2}\,\delta y$

So $\quad V = \int_{-1}^{1} 12\pi\sqrt{1-y^2}\,dy$

$= 12\pi \times \dfrac{\pi}{2}$

$\therefore \quad V = 6\pi^2$

Considers that the solid is made of many thin horizontal rings	✓
Determines x_1	✓
Determines x_2	✓
Gives δV in terms of x_1 and x_2 (or y)	✓
Simplifies δV	✓
Writes the integral for V	✓
States the answer	✓

OR

$x_1 = 3 - \sqrt{1-y^2}$ and $x_2 = 3 + \sqrt{1-y^2}$

$V = \int_{-1}^{1} \pi\left(3+\sqrt{1-y^2}\right)^2 dy - \int_{-1}^{1} \pi\left(3-\sqrt{1-y^2}\right)^2 dy$

$= \int_{-1}^{1} 12\pi\sqrt{1-y^2}\,dy$

$= 12\pi \times \dfrac{\pi}{2}$

$= 6\pi^2$

Determines x_1	✓
Determines x_2	✓
States formula for Volume of revolution about y-axis	✓
Writes the volume as the difference of two integrals	✓
Determines the correct bounds	✓
Integrates for V	✓
States answer exactly	✓

Question 20 **(6 marks)**

Consider a **real** coefficient polynomial $P(z) = z^3 - 4z^2 + cz + d$ where c and d are unknowns.

(a) Given that $z = u\ (\neq 0)$ is a root of the equation $z^3 - 4z^2 + cz + d = 0$, determine one solution of the equation $dz^3 + cz^2 - 4z + 1 = 0$ in terms of u. (3 marks)

$z = u$ is a root of $P(z) = 0$

$\Rightarrow \quad u^3 - 4u^2 + cu + d = 0$

$\Rightarrow \quad 1 - \dfrac{4}{u} + \dfrac{c}{u^2} + \dfrac{d}{u^3} = 0$

$\Rightarrow \quad z = \dfrac{1}{u}$ is a root of $Q(z) = 0$

where $Q(z) = 1 - 4z + cz^2 + dz^3$

Sets up the equation	✓
Divides the equation by u^3	✓
States the answer	✓

(b) Given that $z = 1 + i$ is a root of the equation $z^3 - 4z^2 + cz + d = 0$, determine the values of c and d. (3 marks)

$z = 1 + i$ is a root of $P(z) = 0$

$\Rightarrow \quad z = 1 - i$ is a root of $P(z) = 0$

$\Rightarrow \quad (1+i)^3 - 4(1+i)^2 + c(1+i) + d = 0$

and $\quad (1-i)^3 - 4(1-i)^2 + c(1-i) + d = 0$

(Using CAS :)

$\Rightarrow \quad c = 6 \quad$ and $\quad d = -4$

Writes $z = 1 - i$ as another root	✓
Sets up two equations	✓
States the answers	✓

Question 21 **(7 marks)**

The end points of a movable rod AB of length 1 metre have coordinates (x, 0) and (0, y). The position of the end point A on the x-axis is

$$x(t) = \frac{1}{2}\sin\frac{\pi t}{6}$$

where t is the time in seconds.

(a) Determine the period of point B. (2 marks)

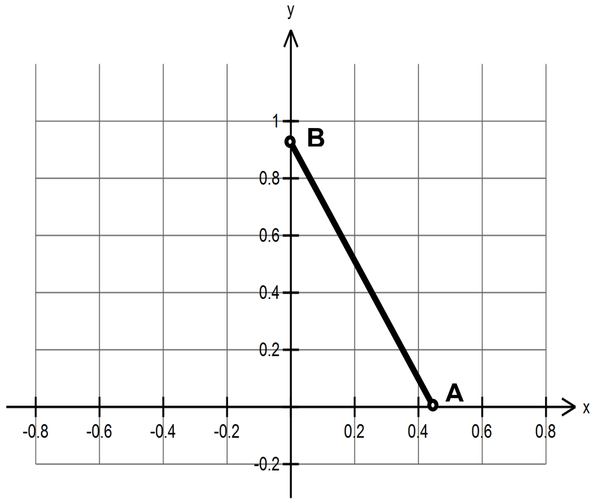
$$x = \frac{1}{2}\sin\frac{\pi t}{6}$$

$$\Rightarrow \quad \text{A's period} = \frac{2\pi}{\pi/6} = 12 \text{ s}$$

$$\Rightarrow \quad \text{B's period} = \frac{12}{2} = 6 \text{ s}$$

Determines A's period ✓

Halves A's period to get B's period ✓



(b) Determine the maximum speed of B and the first time it happens. (5 marks)

$$x = \frac{1}{2}\sin\frac{\pi t}{6} \quad \text{and} \quad x^2 + y^2 = 1$$

$$\Rightarrow \quad y = \sqrt{1 - x^2} = \sqrt{1 - \frac{1}{4}\sin^2\frac{\pi t}{6}}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{1}{2}\,\frac{-\frac{1}{4}\times 2(\frac{\pi}{6})\sin\frac{\pi t}{6}\cos\frac{\pi t}{6}}{\sqrt{1 - \frac{1}{4}\sin^2\frac{\pi t}{6}}}$$

$$\Rightarrow \quad \frac{dy}{dt} = -\frac{\pi}{48}\,\frac{\sin\frac{2\pi t}{6}}{\sqrt{1 - \frac{1}{4}\sin^2\frac{\pi t}{6}}}$$

Using CAS:

$$t = 1.57 \text{ s} \quad \text{and} \quad \max\frac{dy}{dt} = 0.07 \text{ m/s}$$

Sets up the equation $x^2 + y^2 = 1$ ✓

Writes y in terms of t ✓

Expresses dy/dt in terms of t ✓

States the answer for t ✓

States the answer for dy/dt ✓

Additional working space

Question number(s): ________________

Additional working space

Question number(s): ________________

Additional working space

Question number(s): ________________